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Vol. XI BATON ROUGE, LA., APRIL, 1937 No. 7

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This Journal is dedicated to the following aims:

1. Through published standard papers on the culture aspects, humanism and history of mathematics to deepen and to widen public interest in its values.
2. To supply an additional medium for the publication of expository mathematical articles.
3. To promote more scientific methods of teaching mathematics.
4. To publish and to distribute to groups most interested high-class papers of research quality representing all mathematical fields.

The Handwriting on the Wall

Addressed to a thousand teachers of college mathematics and responded to by nearly eight hundred of them, the Seidlin questionnaire (See No. 8, Vol. 10, National Mathematics Magazine) revealed beyond question that among American mathematicians at present is a most discouraging lack of accord, both in the diagnosis of and in the remedies they would provide for the existing situation in secondary mathematics. Certain facts in the situation are denied by none. For those whose life work is the teaching of mathematics, the most dramatically important of these undeniables is the steady decrease of required mathematics in the high school curriculum, as vocational-minded curriculum makers of nearly every state go on building course materials, beset with but little concern for programs aiming at culture or science or the chimera (?) of mental discipline.

The executive policies of the average school supervisor must very largely reflect the hard-headed viewpoint of his school board members, many of whom are likely to be against high school algebra because James and Susan find it too hard, and geometry will never help William to get a job. With democracy in control, it was inevitable that secondary programs should be centered about the trade or the vocation, and, therefore quite unavoidable that only practical and vocational mathematics should have embodiment in a secondary curriculum. Thus is the fading of classic elementary algebra and geometry from the secondary picture mainly traceable to the democratic character of our government, and to an economic pressure that forces into prime importance school programs aiming to prepare youth to make a living.

*Hope of reform would seem vain as increasing hordes of young men and women enter our colleges and universities,—enter with an ignorance so crass that many would have been better off had they had no instruction whatever in beginning algebra and geometry.

In the meanwhile the upper reaches of Science and Engineering demand more and more of mathematics. More and more nearly universal is becoming the call for a technique, furnishable only by mathematics,—a technique that shall substitute precision for inexactness, definitely measured things for things hitherto without measure, the clear-cut formula for cloudy enunciation. Truly a half-science can become a complete one only with the aid of mathematics. The handwriting is on the wall. Our colleges have a responsibility not to be evaded.

*At Louisiana State University during the 1936-37 session 168 freshmen were separated from the regular freshman classes in mathematics, and put into high school mathematics under the classification "Mathematics A", a non-credit course.

On a Cubic Curve Associated With a Triangle

By J. H. WEAVER
Ohio State University

Let there be a triangle ABC and a point P in the plane of the triangle. Draw the lines AP, BP, CP cutting BC, CA, AB respectively in the points A',B',C'. We desire to find the locus of P such that the area of the triangle $A'B'C'$ shall be constant.

Let the point B be the origin of coordinates and BC be along the $x-$axis. We may then write the coordinates of the given points as follows: $A : (a,b)$, $B : (0,0)$, $C : (c,0)$, $P : (x',y')$.

The equation of the line through B and C is

$$y=0, \tag{1}$$

and the equation of the line through A and P is

$$x(y'-b)+y(a-x')=ay'-bx'. \tag{2}$$

From (1) and (2) we find that the co-ordinates of A' are

$$A' : \left(\frac{ay'-bx'}{y'-b}, 0 \right).$$

In a similar manner we find

$$B' : \left(\frac{bcx'}{bx'+cy'-ay'}, \frac{bcy'}{bx'+cy'-ay'} \right)$$

$$C' : \left(\frac{acy'}{ay'+bc-bx'}, \frac{bcy'}{ay'+bc-bx'} \right).$$

If the area of the triangle $A'B'C'$ is mbc we have

$$\begin{vmatrix} \dfrac{ay'-bx'}{y'-b} & 0 & 1 \\ \dfrac{bcx'}{bx'+cy'-ay'} & \dfrac{bcy'}{bx'+cy'-ay'} & 1 \\ \dfrac{acy'}{ay'+bc-bx'} & \dfrac{bcy'}{ay'+bc-bx'} & 1 \end{vmatrix} = 2mbc$$

If we drop primes this reduces to

(3) $y(ay-bx)(bc-cy+ay-bx)-m(y-b)(ay+bc-bx)(bx+cy-ay)=0$

Equation (3) represents a cubic curve. We now proceed to prove the following theorem.

Theorem I. The curve (3) passes through the vertices of the triangle $A\,B\,C$ and its tangents at the vertices are parallel to the opposite sides of the triangle.

Proof: Consider the side $B\,C : y=0$. This side cuts the curve (3) in the finite points (0,0) and $(c,0)$. If we find the expression for the slope of (3) and substitute in this expression the co-ordinates of the points just found, we discover that the slopes are those of the sides of the triangle opposite these points. The same condition obtains at the point (a,b). Hence the theorem.

By means of the usual methods it may be shown that the curve (3) has three real asymptotes, given by the following equations.

$a-c \qquad (a-c)(m+1)$

$$y = \frac{mb}{m+1}$$

(A)
$$y = \frac{bx}{a} - \frac{mbc}{a(m+1)}$$

$$y = \frac{bx}{a-c} - \frac{bc}{(a-c)(m+1)}$$

We may now state the following theorem.

Theorem II. The curve (3) has three real asymptotes which are parallel to the sides of the triangle ABC. These asymptotes will in general form a triangle homothetic to the triangle ABC with the centroid of ABC as homothetic center.

The proof is simple and will be omitted.

Let PQR be the triangle formed by the asymptotes of (3). Then as above it may be shown that the area of PQR is given by

$$\text{Area } PQR = \frac{bc(2m-1)^2}{2(m+1)^2} \tag{4}$$

If in (4) we set $M=\frac{1}{2}$, the area of PQR will be zero. Hence we have the theorem

Theorem III. If the area of $A'B'C'$ is equal to the area of ABC the asymptotes of the curve (3) are concurrent at the centroid of ABC.

For $m=0$ the cubic (3) degenerates into the three sides of the triangle ABC. The asymptotes also coincide with the sides of ABC. The area of PQR from (4) becomes $\frac{1}{2}bc$, which is the area of ABC. If m becomes indefinitely large, the curve (3) and its asymptotes approach the three lines through the vertices of the triangle ABC and parallel to the opposite sides.

In the above discussion m has been assumed to be positive. If now we consider m as a parameter and let it assume the value $m=-1$, the curve (3) degenerates into the line at infinity and the ellipse

$$b^2x^2+(bc-2ab)xy+(a^2-ac+c^2)y^2+bc(ay-cy-bx)=0. \tag{5}$$

The three asymptotes go off to the line at infinity.

The ellipse (5) has its center at the centroid of ABC. It is known as Steiner's ellipse, and is the locus of the isotomic conjugates of the line at infinity.*

If now we write (3) as a quadratic in x we have

$$\begin{aligned} b^2x^2(y+my-mb)+x(y+my-mb)(bcy-2aby-b^2c) \\ +(abcy^2-acy^3+a^2y^3+ma^2y^3 \\ +mabcy^2-macy^3-mbc^2y^2-ma^2by^2-mab^2cy \\ +mabcy^2+mb^2c^2y)=0 \end{aligned} \tag{6}$$

*See Gallatley, Modern Geometry of the Triangle, p. 108. Encyklopädie der Mathematischen Wissenschaften, Band III, Heft 7, p. 1210.

The discriminant of (6) is, after discarding a factor b^2,

$$(7)\quad (cy-2ay-bc)^2(y+my-mb)^2-4[y^3(m+1)(a^2-ac)+y^2(abc+2mabc-mbc^2-ma^2b)+y(mb^2c^2-mab^2c)]\,(y+my-mb)=D.$$

If we set $D=0$ we have a quartic in y from which we obtain the following values for y.

$$(8)\quad y=\frac{mb}{m+1}$$

$$(9)\quad y=b$$

$$(10)\quad y=\frac{b(1-2m)\pm b\sqrt{1-8m}}{2(m+1)}$$

Equations (8) and (9) tell us that the curve (3) is tangent to its asymptote and to a line through A parallel to BC. Equation (10) tells us that the curve (3) is also tangent to two other lines which will be real and distinct if $m<\frac{1}{8}$, real and coincident if $m=\frac{1}{8}$ and imaginary if $m>\frac{1}{8}$.

Hence if $m<\frac{1}{8}$ the curve (3) consists of two branches, one of them a hyperbolic serpentine and the other an elliptic oval. In particular, if $m=-1$, the serpentine branch becomes the line at infinity and the elliptic oval becomes the Steiner ellipse. If $m=\frac{1}{8}$ the curve (3) consists of a hyperbolic sepentine branch and an isolated point. The isolated point is the centroid of ABC. If $m>\frac{1}{8}$ the curve (3) consists of a single hyperbolic serpentine branch. Figures 1, 2 and 3 illustrate the three cases.

Graphical Representation of Complex Roots

By JAMES A. WARD
University of Wisconsin

The purpose of this paper is to show a graphical method of obtaining the imaginary, as well as the real, roots of the cubic and quartic equations. It becomes evident that a single graph is sufficient to solve all cubic equations. Further, the method is extended to equations of higher degree.

1. *Cubic.* The general cubic with real coefficients may be reduced by a linear transformation to the form

$$x^3+px+t=0 \tag{1}$$

Let $u+iv$ be an imaginary root of (1). Then if $u+iv$ be substituted for x in (1), the real and imaginary parts must vanish separately. The imaginary part is

$$iv(-v^2+3u^2+p)=0 \tag{2}$$

Since $u+iv$ is imaginary, v does not vanish.

Therefore

$$-v^2+3u^2+p=0. \tag{3}$$

Conversely, every solution (u, v) of (3) makes $u+iv$ a solution of (1) for some value of t. The real locus (3) is an hyperbola in the complex plane, whose asymptotes are independent of P and are inclined at angles of 60° and 120° with the positive $U-$axis. This hyperbola is the locus of the complex roots of (1) with t as a parameter. Note that if p is negative, the transverse axis of (3) is along the $U-$axis, while if p is positive the transverse axis is along the $V-$axis.

If the graph of $y=-x^3-px$ is drawn, the abscissas of its intersections with the line $y=t$ give the real roots of (1). If r is the only real root, the real part of the imaginary roots will be $-r/2$, because the sum of the roots of (1) is zero. Hence the imaginary roots will be the intersections of the line $u=-r/2$ and the hyperbola (3).

To do this geometrically, the $UV-$plane is placed below the $XY-$plane so that the $V-$axis is colinear with the $Y-$axis and the

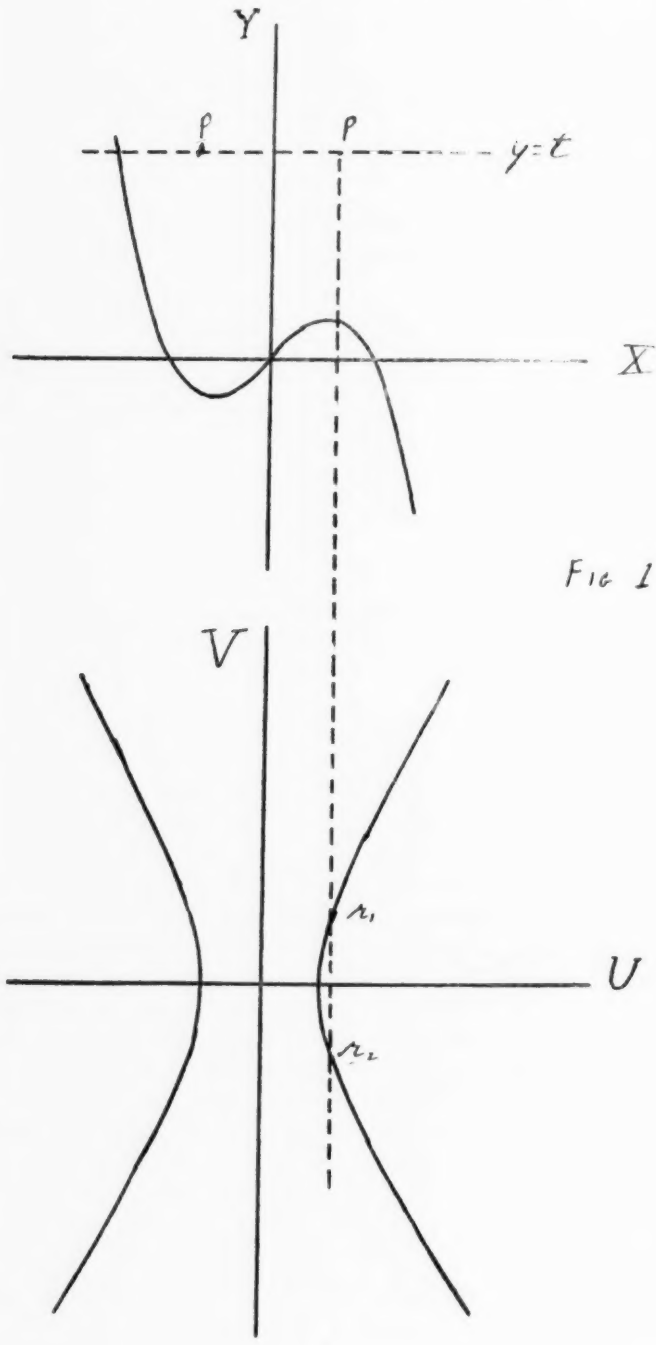

FIG 1

cubic and hyperbola are drawn with the same scale for the complex plane as for the $X-$axis of the real plane (figure 1). If there are three real roots, the line $y=t$ will cut the cubic three times. If there is only one real root, $y=t$ will cut the cubic only once. Let $p(r/2, t)$ be the midpoint of the segment between the cubic and the $Y-$axis. Then $P'(-r/2, t)$ is the point symmetric to it with respect to the $Y-$axis. A vertical line through P' extended into the $UV-$plane will be the line $u=-r/2$ and will intersect the hyperbola in the imaginary roots r_1 and r_2 of (1).

By using only figure 1 for $p<0$ and the conjugate figure for $p>0$, it is possible to solve any cubic geometrically by properly adjusting the scale. If $p<0$ in (1), figure 1 is used and the scale on the $X-$axis is adjusted so that the cubic intersects the $X-$axis at $\pm\sqrt{-p}$ (the complex plane uses the same scale). If $p>0$ the ordinates of the vertices of the hyperbola in the complex plane are $\pm\sqrt{p}$. (If $p=0$, the asymptotes are used.) The scale on the $Y-$axis may be determined by finding any non-zero ordinate. Then from the intersections of the curve and the line $y=t$, the roots may be located as before.

Note that if p is positive, the slope of the cubic is everywhere negative, there is always exactly one real root, and the maximum and minimum of the cubic are imaginary. The transverse axis of the hyperbola (3) is along the $V-$axis, and the two imaginary roots are on different branches of the hyperbola.

If p is negative (see figure 1), the cubic has a real maximum and minimum and the transverse axis of the hyperbola (3) is along the $U-$axis. Let M be the ordinate of the maximum. Then $-M$ is the ordinate of the minimum. If $t>M$, there is one real root and the two imaginary roots are on the positive branch of the hyperbola. As t approaches M, these two imaginary roots approach the vertex of the hyperbola $(\sqrt{-p/3}, 0)$. When $t=M$, the equation has only real roots, two of which are equal. For $-M<t<M$ the roots are all real and different. For $t=-M$, two of the roots are again equal, and for $t<-M$ the imaginary roots are on the negative branch of the hyperbola.

2. *Quartic.* The reduced quartic with real coefficients is

$$x^4+px^2+qx+t=0 \tag{4}$$

Let $u+iv$ be an imaginary root. Then as in the case of the cubic the vanishing of the imaginary part (after dividing by iv) leads to

$$4uv^2=4u^3+2pu+q \tag{5}$$

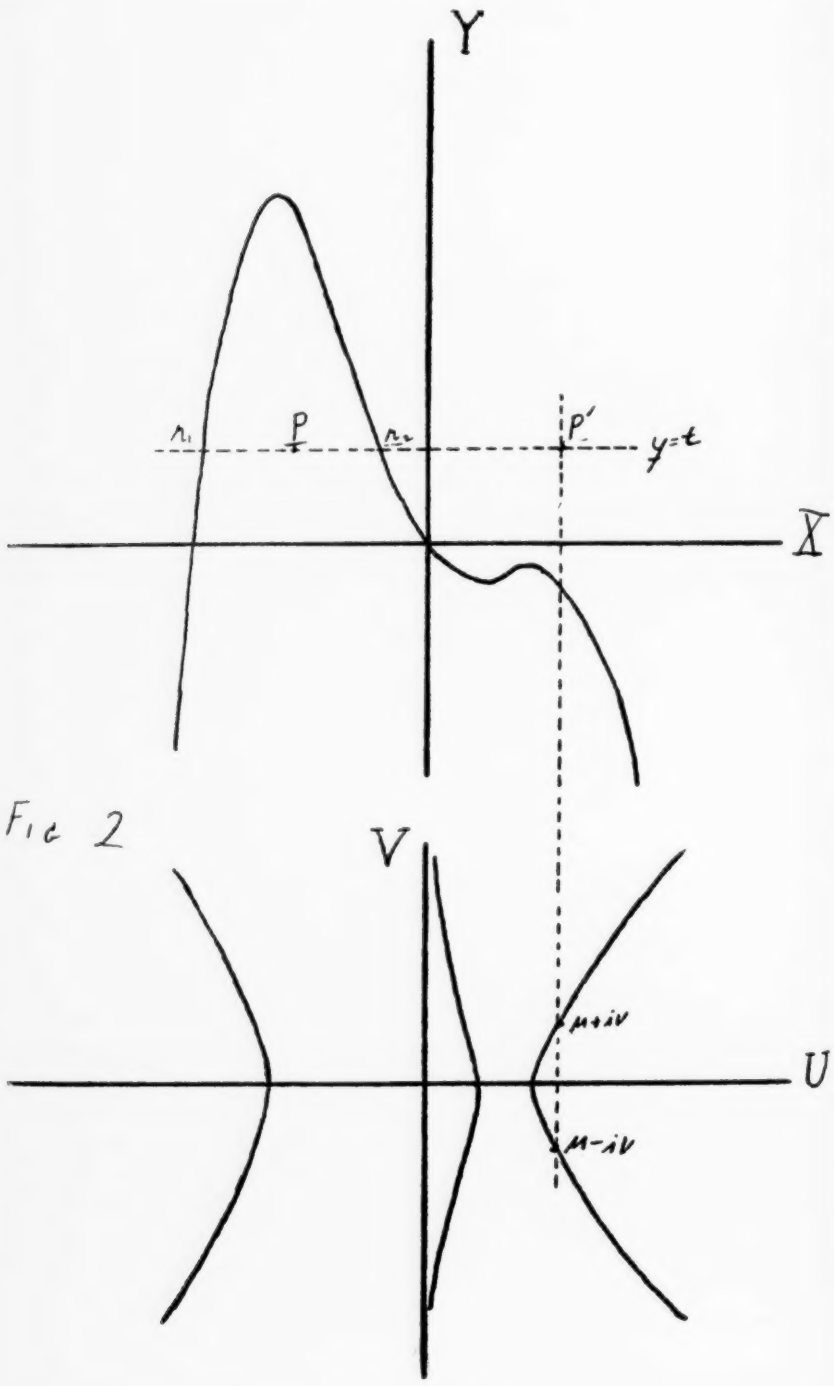

Fig 2

which is the locus of the imaginary roots of (4). The graph of (5) has three fixed asymptotes

$$u=0,\ u=v,\ u=-v$$

which are independent of p and q and hence there are three distinct types of loci.

If the graph of $y=-x^4-px^2-qx$ is drawn, the abscissas of its intersections with the line $y=t$ will give the real roots of (4) for any real value t. If there are 4 real roots, there will be 4 intersections. If there are exactly 2 real roots r_1 and r_2, the real part of the complex roots is $-\frac{1}{2}(r_1+r_2)$ because $2u+r_1+r_2=0$ from (4). Therefore the roots will be the intersection of the line $u=-\frac{1}{2}(r_1+r_2)$ with the curve (5).

For geometrical location of the complex roots, the quartic and the curve (5) are drawn (figure 2) with axes placed as for the cubic. Since $y=t$ cuts the quartic only twice, we let P be the midpoint of the intersections and P' the point symmetric to it with respect to the $Y-$axis. A vertical line through P' extended into the $UV-$plane will intersect the curve (5) at 2 points which are the imaginary roots of (4).

If no real roots exist, the line $y=t$ will not intersect the quartic at all. Therefore it will be necessary to draw an auxiliary curve. After substituting $u+iv$ in (4) the vanishing of the real part leads to

$$v^4-v^2(6u^2+p)+u^4+pu^2+qu+t=0.$$

By eliminating v from this and (5), we obtain

$$64u^6+32pu^4+4(p^2-4t)u^2-q^2=0. \tag{6}$$

This is the resolvent cubic in u^2 which is used in Descartes' solution of the quartic*, and therefore may be written down immediately. Therefore

$$t=\frac{64u^6+32pu^4+4p^2u^2-q^2}{16u^2} \tag{7}$$

This is plotted on the real plane. Since it is the locus of the real part of the imaginary roots of the quartic, its intersections with $y=t$ are the points through which vertical lines are drawn and extended into the $UV-$plane to determine the imaginary roots (figure 3). The locus (7) contains all the points P', hence this scheme may be used if there are 2 real roots. Figures 2 and 3 are for the quartic

*Dickson: *"First Course in the Theory of Equations,"* page 52, equation 22.

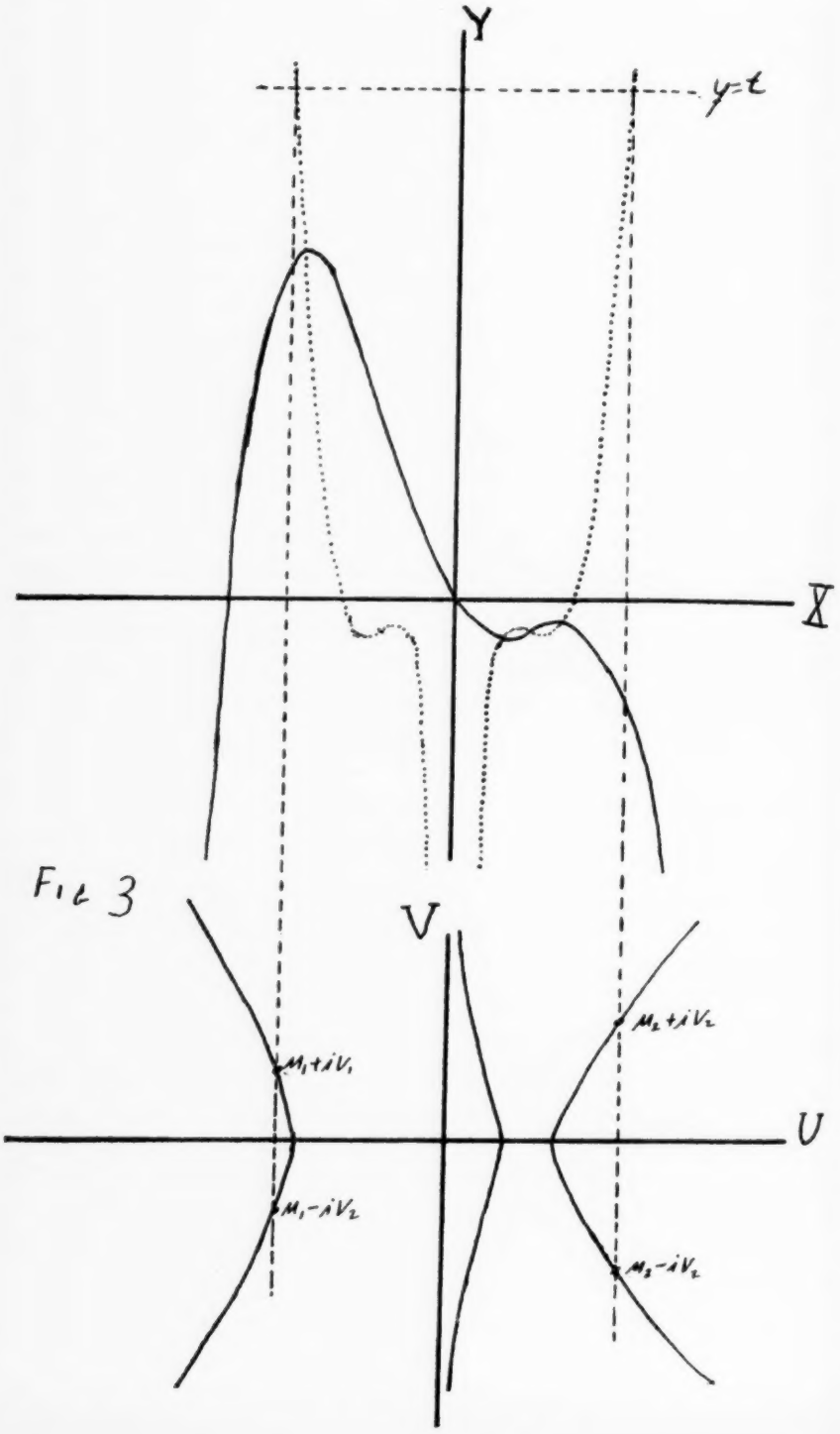

FIG 3

The special case in which $q=0$ in (4) may most easily be done by elementary algebra. The graphical method must be treated separately, for the above does not give a method for finding the pure imaginary roots of (4) in all cases. In this instance (5) may be written

$$u(-2v^2+2u^2+p)=0$$

Hence one part of its graph is $u=0$. The auxiliary curve, (6), may be written similarly and its graph consists of the line $u=0$ and a quartic. Hence the line $y=t$ will always intersect the auxiliary curve at some P' on $u=0$. When this is projected to the $UV-$plane, it determines the line $u=0$, the totality of all pure imaginary points. In order to find which, if any, of these are solutions of (4) it is necessary to use an additional equation. The real part of (4), obtained by substituting $x=u+iv$ and setting $u=0$ (also remember $q=0$), is

$$v^4-pv^2+t=0. \tag{8}$$

Since v is restricted to real values, the real roots of (8) are the pure imaginary roots of (4), and may be found by the ordinary graphical method. (8) may have four, two, or no real roots. In any case the remaining roots of (4) may be found by the method given in the preceeding paragraphs.

$$x^4-56x^2+192x+t=0.$$

3. *Higher degree equations* of degree n with real coefficients of the form $f(x)=0$ can be solved the same way. Let one of the complex roots be $u+iv$ and the imaginary part after dividing out iv (which is non-vanishing) becomes

$$\sum_{j=1}^{n}(-1)^j\frac{v^{2j-2}f^{(2j-1)}(u)}{(2j-1)!}=0$$

which is the locus of the complex roots. However if there is more than one pair of conjugate imaginary roots, it is necessary to have the auxiliary equation.

Two Families of Parallelograms

By KARLETON W. CRAIN
Purdue University

Theorem I.

If in a convex quadrilateral the diagonals be drawn, they divide its area into four triangles, and the four points which divide into a constant ratio the line segments joining the circumcenters of these four triangles with the corresponding orthocenters, determine the vertices of a parallelogram.

By their construction we see that the circumcenters of these four triangles determine the vertices of a parallelogram; likewise the four orthocenters determine the vertices of another parallelogram. (Cf. Figure 1.) Since the points which divide into the same constant ratio

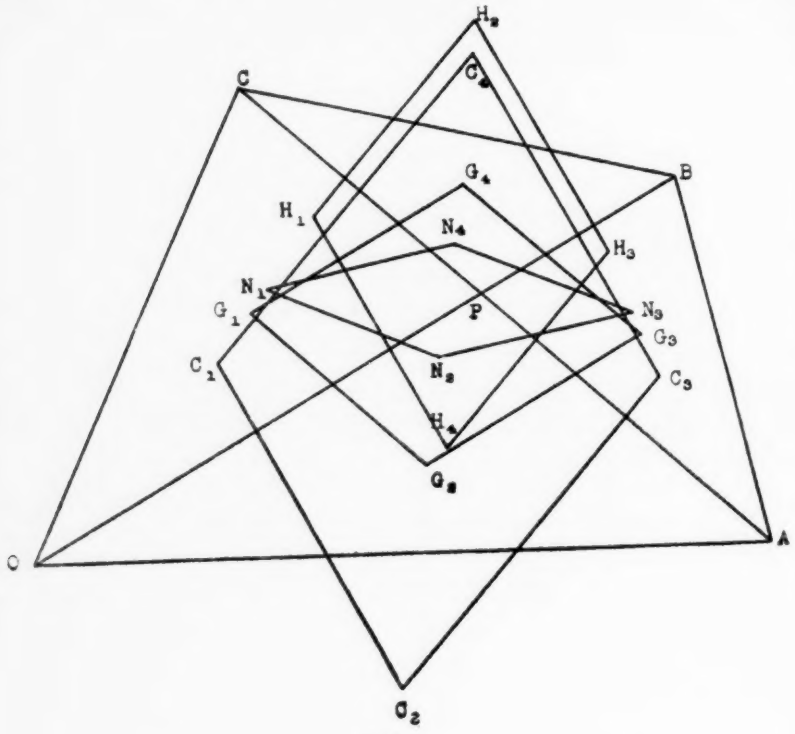

FIG. 1

the line segments which connect the vertices of one parallelogram to the corresponding vertices of another parallelogram are the vertices of a third parallelogram,* theorem I is established.

For each constant ratio taken, there is such a parallelogram. Therefore, there is a one-parameter family of parallelograms associated with these two.

In particular, the four centroids of these triangles determine a parallelogram, since the centroid divides the line segment joining the circumcenter and the orthocenter into the ratio one to two. Also the four Nine Point Centers of these triangles are the vertices of another parallelogram, since the N. P. C. bisects the line segment joining the circumcenter and the orthocenter. It is interesting to note that these lines joining the circumcenters with their corresponding orthocenters are the Euler lines of the four triangles.

Let the vertices of the quadrilateral have the rectangular coordinates $O(0,0)$, $A(x_2, y_2)$, $B(x_1\ 0)$ and $C(x_3, y_3)$. Let H_1, H_2, H_3, H_4, be the orthocenters of triangles OPC, APO, BPA, and CPB respectively. Then the equations of H_1H_2, H_2H_3, H_3H_4, H_4H_1 are

$$x(x_3-x_2)+y(y_3-y_2)=0.$$

$$x=x_2.$$

$$x(x_3-x_2)+y(y_3-y_2)+x_1x_2-x_1x_3=0.$$

$$x=x_3.$$

Solving these equations simultaneously in pairs, the coordinates of the orthocenters are found to be,

$H_1[x_3, x_3(x_2-x_3)/(y_3-y_2)]$, $H_2[x_2, x_2(x_2-x_3)/(y_3-y_2)]$,

$H_3[x_2, (x_3-x_2)(x_1-x_2)/(y_3-y_2)]$, and

$H_4[x_3, (x_1-x_3)(x_3-x_2)/(y_3-y_2)]$. Hence the area of

$H_1H_2H_3H_4$ is $x_1(x_3-x_2)^2/(y_3-y_2)$.

Again if C_1, C_2, C_3, C_4 are the circumcenters of triangles OPC, APO, BPA, and CPB respectively, the equations of C_1C_2, C_2C_3, C_3C_4, C_4C_1 are

$$x=(x_2y_3-x_3y_2)/2(y_3-y_2).$$

$$2x(x_3-x_2)(y_3-y_2)+2y(y_3-y_2)^2-y_3(y_3-y_2)^2 + (x_3-x_2)(x_2y_3+x_3y_2-2x_2y_3)=0.$$

*Mathesis, 1907, page 241.

$$x=(x_2y_3-x_3y_2+x_1y_3-x_1y_2)/2(y_3-y_2).$$

$$2x(x_3-x_2)(y_3-y_2)+2y(y_3-y_2)^2-y_3(y_3-y_2)^2 +(x_3-x_2)(2x_3y_2-x_2y_3-x_3y_3)=0.$$

Solving these equations simultaneously in pairs, the coordinates of the circumcenters are,

$$C_1\left[\frac{x_2y_3-x_3y_2}{2(y_3-y_2)},\ \frac{y_3(y_3-y_2)+x_3(x_3-x_2)}{2(y_3-y_2)}\right],$$

$$C_2\left[\frac{x_2y_3-x_3y_2}{2(y_3-y_2)},\ \frac{x_2(x_3-x_2)+y_2(y_3-y_2)}{2(y_3-y_2)}\right],$$

$$C_3\left[\frac{x_2y_3-x_3y_2+x_1y_3-x_1y_2}{2(y_3-y_2)},\ \frac{y_2(y_3-y_2)+(x_2-x_1)(x_3-x_2)}{2(y_3-y_2)}\right],$$

$$C_4\left[\frac{x_2y_3-x_3y_2+x_1y_3-x_1y_2}{2(y_3-y_2)},\ \frac{y_3(y_3-y_2)+(x_3-x_1)(x_3-x_2)}{2(y_3-y_2)}\right]$$

Hence the area of $C_1\,C_2\,C_3\,C_4$ is

$$\frac{x_1[(x_3-x_2)^2+(y_3-y_2)^2]}{4(y_3-y_2)}$$

Using the point of division formula, we find that the centroids of the triangles OPC, APO, BPA, and CPB are

$$G_1\left[\frac{x_3y_3-2x_3y_2+x_2y_3}{3(y_3-y_2)},\ \frac{y_3}{3}\right],\quad G_2\left[\frac{2x_2y_3-x_2y_2-x_3y_2}{3(y_3-y_2)},\ \frac{y_2}{3}\right],$$

$$G_3\left[\frac{2x_2y_3-x_2y_2-x_3y_2+x_1y_3-x_1y_2}{3(y_3-y_2)},\ \frac{y_2}{3}\right],$$

$$G_4\left[\frac{x_3y_3-2x_3y_2+x_3y_3+x_1y_3-x_1y_2}{3(y_3-y_2\ \ )},\ \frac{y_3}{3}\right]$$

The area of $G_1\,G_2\,G_3\,G_4$ is

$$\frac{x_1(y_3-y_2)}{9}.$$

Using the midpoint formula, the Nine Point Centers of triangles *OPC, APO, BPA, CPB* are

$$N_1\left[\frac{2x_3y_3-3x_3y_2+x_2y_3}{4(y_3-y_2)}, \frac{x_3(x_2-x_3)+y_3(y_3-y_2)}{4(y_3-y_2)}\right],$$

$$N_2\left[\frac{3x_2y_3-2x_2y_2-x_3y_2}{4(y_3-y_2)}, \frac{x_2(x_2-x_3)+y_2(y_3-y_2)}{4(y_3-y_2)}\right],$$

$$N_3\left[\frac{3x_2y_3-2x_2y_2-x_3y_2+x_1y_3-x_1y_2}{4(y_3-y_2)}, \frac{(x_3-x_2)(x_1-x_2)+y_2(y_3-y_2)}{4(y_3-y_2)}\right],$$

$$N_4\left[\frac{2x_3y_3-3x_3y_2+x_2y_3+x_1y_3-x_1y_2}{4(y_3-y_2)}, \frac{(x_1-x_3)(x_3-x_2)+y_3(y_3-y_2)}{(4y_3-y_2)}\right].$$

The area of $N_1N_2N_3N_4$ is

$$\frac{x_1[(y_3-y_2)^2-3(x_3-x_2)^2]}{16(y_3-y_2)}$$

The area of $OABC$ is $x_1(y_3-y_2)/2$.

If we denote the areas of the four parallelograms determined by the centroids, the circumcenters, the orthocenters, and the Nine Point Centers by G, C, H, and N respectively, and the area of the quadrilateral $OABC$ by Q, we have the following unique relations among these several areas,

$C=Q/2+H/4.$ $Q=9G/2.$ $C=9G/4+H/4.$

$N=9G/16-3H/16.$ $N=Q/8-3H/16.$

$N=Q/2-3C/4.$ $N=9G/4-3C/4.$

$C=Q/2-H/4.$ $N=Q/8-9H/16.$ $Q=9G/2.$ $N=9C/4-Q.$

$C=9G/4-H/4.$ $N=9(G-H)/16.$ $N=9C/4-9G/2.$

Theorem II.

In a convex quadrilateral the diagonals divide its area into four triangles; if two lines are drawn through the intersection of the diagonals so that one line cuts the circumcircles of two opposite triangles,

and the other line cuts the circumcircles of the other two opposite triangles in points which are outside the quadrilateral, then the four Simpson lines of these points with respect to their corresponding triangles determine a parallelogram whose angles are equal to the angles formed by these two lines. (Cf. Figure 2)

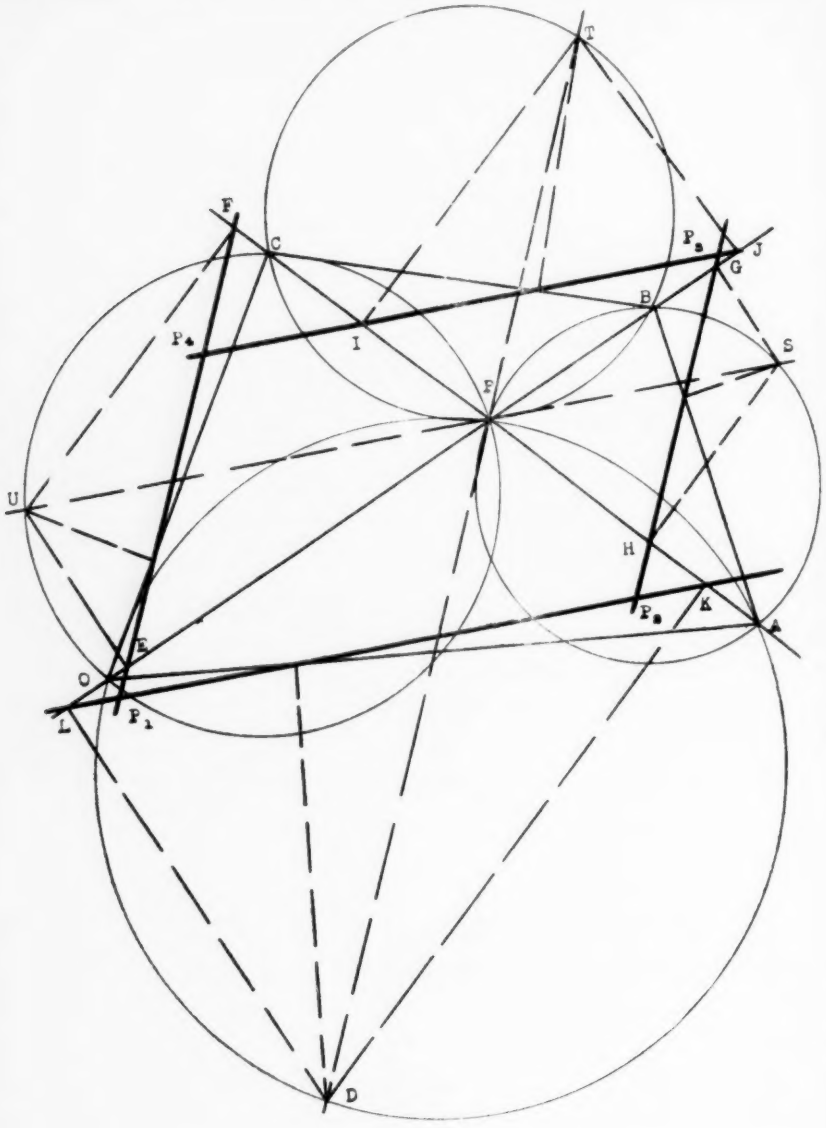

FIG. 2

In the two inversely similar quadrilaterals $DKPL$ and $PGSH$, $\angle PDL = \angle SPH$, but the points P, L, D, K lie on a circle whose diameter is DP and $\angle PDL = \angle LKP$. Therefore, $\angle LKP = \angle SPH$ and LK is parallel to US. Similarly, IJ is parallel to US. Also EF and HG are parallel to DT. Therefore, the figure $P_1P_2P_3P_4$ is a parallelogram whose angles are equal to the angles formed by the lines DT and US.

This group of parallelograms thus is a two-parameter family, as each of the lines DT and US is chosen independently.

Pestalozzi and the American Arithmetic*

By E. R. SLEIGHT
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Out of poverty very frequently comes a suggestion that leads to important results. Such seems to have been the case in the system of education known as the Pestalozzi system, originated by Johann Heinrich Pestalozzi who was born January 12th, 1746, in Zurich, Switzerland. During his university course he belonged to an association of young men known as the Patriots. They interested themselves in redressing the grievances of the poor and oppressed. His interest in the welfare of destitute children prompted him in 1775 to establish a school in which such children could be educated free of charge. This school continued for five years only, but he did not give up his dream. In 1805 he was given an old castle, and he began his last effort to prove the efficiency of his system. Assistants were engaged, and the fame of his school spread abroad. Those who could afford to pay were asked to do so. This effort was a decided success. He insisted that children should be led step by step to a complete understanding and not be permitted to memorize only. The idea was received very enthusiastically by educators, and we find its influence causing a very marked change in the content and plan of the American Arithmetics.

Daniel Adams, whose earlier Arithmetic was very warmly received, re-wrote his text in 1823, introducing the analytic method of Pestalozzi. But in the same year, another arithmetic appeared which seems more definitely to have caught the spirit of the great educator. This little book was written by Warren Colburn, and was called *"First Lessons in Arithmetic on the Plan of Pestalozzi with Some Improvements."* Two

*Read at Michigan Section of Mathematical Association, Ann Arbor, Michigan, March 20, 1937.

very strong recommendations are found. One of them by J. Farrar, Professor of Mathematics, Harvard University, reads as follows: "Having been made acquainted with Mr. Colburn's treatise on Arithmetic, and having attended an examination of his scholars, who had been taught according to this system, I am well satisfied that it is the most easy, simple, and natural way of introducing young persons to the first principles in the science of numbers. This little work is therefore earnestly recommended to the notice of those who are employed in this branch of early instruction, with the belief that it only requires a fair trial to be fully approved and adopted."

In following the Pestalozzi system three features stand out preeminently:

1. All operations proceed from the concrete to the abstract "in a very natural manner."

2. Processes are illustrated by very simple examples, "entirely within the comprehension of the child."

3. The use of plates "as an aid to the understanding."

The author in his preface states "We first make calculations about sensible objects, and we soon observe that the same calculations will apply to very dissimilar things, and finally that they may be made to apply without reference to any particular things. Hence from particulars we establish general principles." That Colburn very carefully carried out this plan is easily observed when one reviews the book. He caught the idea that abstracts are of very little use until the learner has discovered the principle of practical examples. He then makes use of abstract examples to fix the principles. This book is a complete change from any other text previously written, which I have examined. In using Colburn's own words, "The usual method of procedure is as follows: The pupil learns a rule, which to the man who wrote it, was a general principle; but to him, and often to the instructor, it is so far from it that it hardly deserves to be called even a mechanical device. He is next to apply his rule to a practical example, but if he did not find the example under the rule he would never mistrust that it belonged to it. When an example is proposed to him, which is not in the book, his sagacity is exercised, not in discovering the operation necessary to its solution, but in comparing it with the examples which he has performed before, and in endeavoring to discover some analogy, either in the sound or in something else."

The text, containing only 123 very small pages, is divided into 15 sections. The first three sections, covering more than one-third of the entire material of the book, are devoted to the four fundamental

operations. These are not presented as separate operations, but rather they are all based upon the operation of addition. For, as the author states "It is remarkable that a child, although he is able to perform a variety of examples which involve addition, subtraction, multiplication and division, recognizes no operation but addition."

Fractions are introduced, not as a separate section in the text, but as a result of division. On the same page with many simple examples in division we find the remark, "When anything or any number is divided into equal parts, one part is called the half of the number or thing." The following examples, illustrating this idea, are found immediately after the above definition:

1. If an apple is worth two cents, what will one-half of it be worth?

2. If you can buy one pear for two cents, how many can you buy for three cents?

Definitions for one-third and one-fourth follow, after which are found several pages of simple examples "of the practical type," such as:

"If you can buy a barrel of cider for three dollars, how much can you buy for four dollars? How much for five dollars?"

After this very simple introduction to fractions, the author states his plan, namely, "Every combination commences with practical examples. The numbers are small and the questions are so simple that almost any child of five or six years of age is capable of understanding more than half the book, and those of seven or eight can understand the whole of it." This same informal method serves to introduce all the operations on fractions. As an illustration we find that multiplication comes into the picture by the following example: "If a meal for one man costs one-third of a dollar, what will be the cost for two men?" There is no attempt to introduce a subject as a separate unity, but rather as an example of some operation already known. Interest is introduced as a problem in multiplication, by such examples as "If a man pays six cents for the use of one dollar for a year, what will he pay for the use of five dollars?"

Except for numbering the pages and the examples, symbols for numbers are not used until we arrive at page 38. Here Colburn informs the pupil that "Instead of writing names of numbers, it is usual to express them by particular characters called figures."

There is no doubt but that Colburn possessed a keen insight into the principles and methods of teaching for he makes this significant statement, "Where one finds a difficulty in solving a question, he will remove it much sooner and much more effectively by taking a very

small example of the same kind, and observe how he does it, than by recurring to the rule." He also suggests that "A very useful mode of recitation is for the instructor to read the example to the whole class, and then allowing sufficient time for them to perform the question, call upon someone to answer it. In this way every pupil will be obliged to perform the example, because they do not know who is to answer it."

Following the plan of proceeding from the very simple to the more difficult, the examples are so arranged that several depend upon each other, so that the preceding explains the following one. Sometimes also in the same example there are several questions asked so as to lead the pupil from the simple to the more complex.

One of the most interesting features of Colburn's text is found in the plates which he uses as an aid to the pupil. There are three of them, based on the material in the text. The first plate viewed horizontally, presents ten rows of rectangles, and in each row there are ten rectangles. Each rectangle in the first row contains one mark, each mark representing unity. In the second row, each rectangle contains two marks, in the third three marks, etc. "The purpose of this plate is, first to represent unity, either as a unit, or as making a part of a sum of units. Secondarily, to represent a collection of units, either as forming a unit itself, or as making a part of another collection of units; and thus to compare unity, and each collection of units, with another unit, in order to ascertain their ratios." The last 32 pages of the text make up Part II, in the form of a key, which "contains an

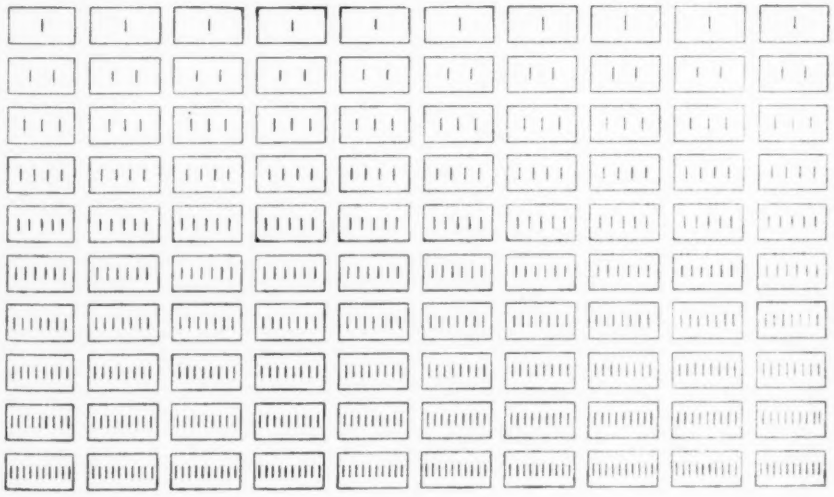

PLATE I

explanation of the plates and the manner of using them." The key is intended primarily for the teacher, and "Many remarks with regard to the manner of illustrating the principles to the pupils are inserted in their proper places." The author also suggests that "Instructors who may never have attended to fractions need not be afraid to undertake to teach this book. The author flatters himself that the principles are so illustrated and the processes are made so simple that anyone who shall undertake to teach it, will find himself familiar with fractions before he is aware of it."

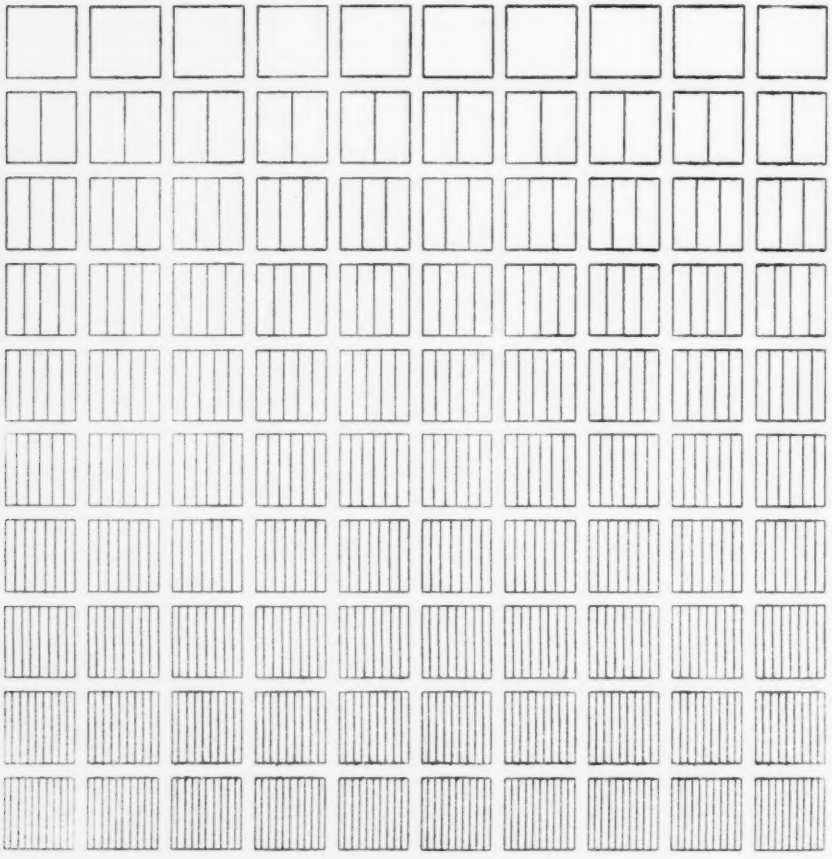

PLATE II

All of the examples in the text may be solved by use of these three plates. Plate I covers the first seven sections, which include the four fundamental operations and very simple fractions. The following example will serve as an illustration of the use of this plate.

"If a man travels 3 miles in one hour, how far will he travel in 4 hours?" To solve this turn to the third row of rectangles, and count the number of marks in the first four rectangles.

Section IV introduces fractions in a very simple manner in such examples as, "At 3 dollars a yard, what will four yards and $\frac{1}{3}$ of a yard of cloth cost?" The third row of Plate I may again be used for the solution of this problem. Count the number of marks in four of the rectangles, and to this result add one mark in the fifth.

Plate I presents each unit as a simple object and undivided. But plate II presents these objects as divisible objects, the different fractions of which form parts and sums of parts of unity. This plate is divided into 10 rows of equal squares, and each row into 10 squares. The first row is composed of 10 empty squares, which are to be presented as entire units. The second row presents 10 squares, each divided into 2 equal parts by a vertical line, and each of these parts is called one-half. In the third row each square is divided into three equal parts by means of two vertical lines, each part representing one-third. This same process is continued to the tenth row, which is divided into 10 equal parts by the use of 9 vertical lines, each part representing a tenth. In this plate as well as the plate that follows, the spaces are to be counted and not the lines. Section VIII to section XII inclusive may be solved by the use of Plate II. Any problem involving fractions may be solved by Plate I, provided only whole units are involved. Fractions involving a part of a unit may be solved by Plate II. For example, "A boy divides four oranges among his companions giving them $\frac{1}{3}$ of an orange apiece. To how many boys did he give them?" If the pupil counts the number of spaces in four squares of the second row, he will have the answer to this problem. To find the product of five times two-thirds, turn to the third row, take the sum of the spaces in three squares together with one space in the fourth square.

Plate III is intended to represent fractions of unity divided into other fractions. It is, therefore, an extension of Plate II. It differs from it in that horizontal, as well as vertical, lines are used, thus dividing the fractions of squares into fractions of fractions. As in Plate II there are 10 rows of squares, each row containing ten equal squares. In the first row the first square is undivided. The remaining squares are divided by horizontal lines into from two to ten equal parts, respectively. In all the other squares the vertical divisions are the same as in Plate II, but in addition each row is divided horizontally in the same manner as the first row. By this means the second row represents fractions from halves to twentieths, the third row from

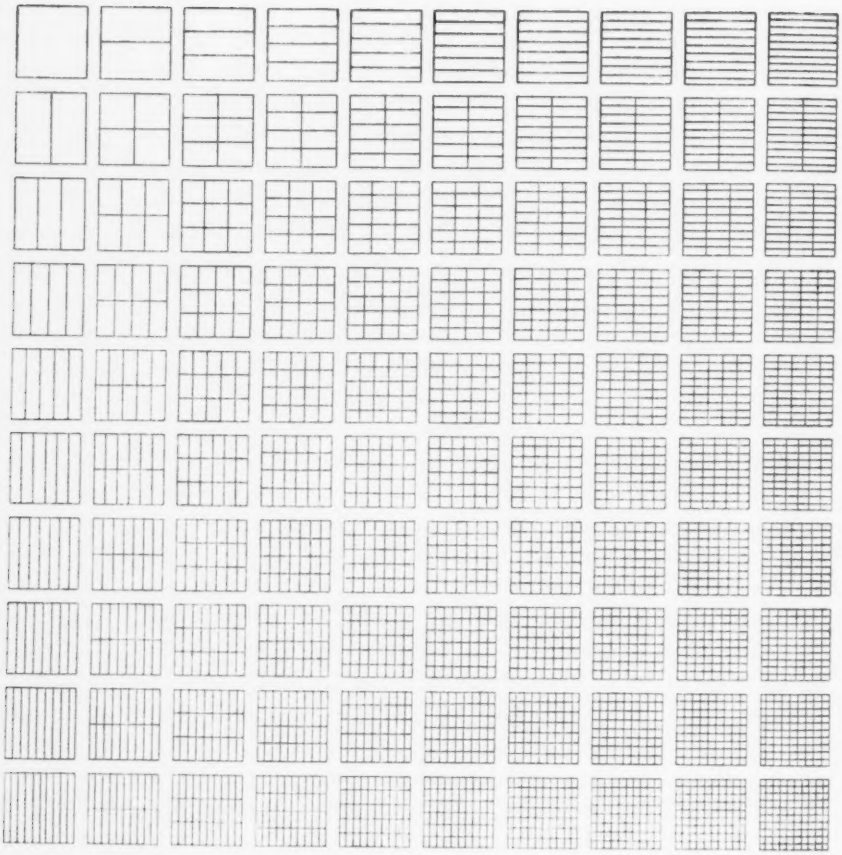

PLATE III

thirds to thirtieths, this plan continuing until we arrive at the tenth row which represents tenths to one hundredths. It is also to be noticed that the second row, in addition to halves, fourths, etc., also represents halves of halves, thirds of halves, etc., while the third row represents, not only thirds, sixths, etc., but also halves of thirds, thirds of thirds, etc. The other rows present similar representations. This plate is used in reducing fractions to common denominators, and in solving problems in the fundamental operations of fractions. The following is the first example in the text to be worked by this plate. "If you give $1/2$ of an orange to one boy, and $1/4$ to another, how much more do you give to the first boy than to the second?" The second square of the second row explains this process. The vertical line divides the square into halves, and the horizontal into fourths. It is easily seen

that $\frac{1}{2}$ is equivalent to $\frac{2}{4}$, and that, therefore, the first had twice as much as the second. If it is proposed to add $\frac{1}{2}$ and $\frac{1}{4}$ this same square may be used. The sum is easily seen to be $\frac{3}{4}$.

An example involving multiplication of fractions proposed by the author is thus stated: "A boy having $\frac{1}{3}$ of an orange gave away $\frac{1}{3}$ of that. What part of the whole orange did he give away?" The third square on the second line is used in this case. This square is divided into nine equal parts of which the boy first had three and then gave away one. He therefore gave away $\frac{1}{9}$ of the entire orange.

There is no attempt to present difficult examples but rather to cover the fundamentals of arithmetic in a variety of ways. Neither do we find the "pleasant and diverting" problems so prevalent in the texts of those days. The aim of the author, as expressed in the introduction, is well carried out. "It is not too bold an assertion to say that no man ever actually learned mathematics in any other method than by analytic induction; that is, by learning the principles by the examples he performs, and not by learning principles first, and then discover by them how examples are to be worked."

Colburn gives Pestalozzi credit for the two outstanding ideas, namely, the method of analytic induction and the plates. But he does claim some credit for himself and closes his introduction in the following words: "In forming and arranging the several combinations, the author has received considerable assistance from the system of Pestalozzi. He has not, however, had an opportunity of seeing Pestalozzi's own work on the subject, but only a brief outline of it by another. In selecting and arranging the examples to illustrate these combinations, and in the manner of solving questions generally, he has received no assistance."

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B. F. Thibaut [1775-1832], Early Master of the Art of Teaching and Popularizing Mathematics

By G. WALDO DUNNINGTON
University of Illinois

Bernhard Friedrich Thibaut was born December 22, 1775, in Harburg, Germany. His father, a Hanoverian army officer, was descended from a Huguenot refugee family, and his mother was a daughter of U. Grupe, a well known legal scholar, Germanist, and journalist. This combination of French and German qualities produced in B. F. Thibaut an extraordinary personality. He was a brother of the famous Heidelberg jurist, Anton Friedrich Justus Thibaut (1772-1840), who after 1810 attracted students from all Germany, and who also accomplished notable work in the field of music and its history. The two brothers went in 1792 to study at the University of Göttingen. Bernhard Friedrich studied mathematics there under Kästner, Beckmann, and Lichtenberg; his manner of lecturing is attributed to their influence. In 1797 he became "Magister and Privatdozent", in 1802 associate professor of mathematics, and in 1805 full professor. In 1819 he received the title of Hofrat.

Thibaut's promotion to an associate professorship in 1802 followed the publication of his *Grundriss der reinen Mathematik* (1801) which reached its fifth edition in 1831. His second work, the *Grundriss der allgemeinen Arithmetik oder Analysis zum Gebrauch bei akademischen Vorlesungen*, Göttingen 1809, 2nd edition: 1830, is historically important. His fame, however, rests not on these two textbooks, but rather on his elegant lectures and kindly human qualities, which inspired many students. Thibaut's offerings were: 1) "a theoretical course, comprising pure mathematics, the analysis of finite magnitudes, and differential and integral calculus, 2) "a practical course, embracing applied mathematics and practical geometry, 3) "higher geometry, 4) "higher mechanics." In 1829 he received the "nominal" professorship of mathematics at Göttingen, and in 1831 became a member of the scientific examination committee for schoolmaster candidates. His death occurred November 4, 1832.

B. F. THIBAUT

traditional elementary instruction for this purpose was obstructive rather than helpful, combined with the transparent lucidity with which fundamental mathematical concepts were at once explained, encouraged everyone during the first lesson to a beneficial faith in his own powers, as well as in the dignity of the subject. Like the others who heard Thibaut for the first time, I left the lecture room intoxicated by the impression of this euphony, this art and lucidity of speaking. But I could not be persuaded that Euclid and Legendre would be entirely without value to me, that indeed according to this teacher's reasoning they would be in my way."

Tellkampf soon learned that this last remark was to be taken *cum grano salis.* Thibaut's lectures were attended by students of all faculties and departments, and this remark was intended for legal and theological students who were studying mathematics without much preparation, in order "to train their reasoning powers" in one semester. Tellkampf continues:

"He emphasized the principle of continuity in sharply expressed opposition to the disconnectedness and apparent arbitrariness of Euclid and his followers, and claimed the fundamental concept of higher geometry for elementary geometry, also. Demonstrating in the simultaneous variability of the components of a figure its logical connection and interpreting in such a form the otherwise fossilized theorems as moments of such a flowing presentation, he knew how to captivate the gifted no less than the beginner by the subject, which appeared in this treatment new and attractive to each one. . . .

"His vigor in mastering and unfolding his material, his sharpness in uncovering the weakness and obscurity in the usual modes of presentation were meanwhile brought even more forcibly to my attention, when during the following days of his course on differential and integral calculus (superficial knowledge of which I had previously gotten only from French textbooks) he explained the modes of presentation on which the inventors of this calculus and their followers sought to establish it. It gave me great satisfaction to see drawn to the light of day here, with a critique, all the doubts as to the simultaneous existence and non-existence of increments, the real meaning of 0/0—in short, as to all those ideas which were always in my mind like silent objections to the entire theory; and Carnot with his *Réflexions sur la métaphysique du calcul infinitesimal,* which seeks rescue in an abolition of errors, ceased to disturb me. The only thing which seriously annoyed me was the slight recognition which Thibaut gave to Leibniz in contrast to Newton, whose concept of fluxions certainly corresponded closely to his above mentioned principle of making fluent the geomet-

rical presentations—the fact that he had to give him preference over the differential calculus of our great countryman.

"But even more completely than in the course mentioned here, did I learn to appreciate Thibaut's merits as a teacher in a later one—on general arithmetic." Tellkampf praises his thoroughness, his patience in complicated calculations, his broad view of mathematical science, and his excellent insight. He contrasts Thibaut's style as a lecturer with his laconicism as an author of textbooks, and explains thus:

"That which lent a very special charm to this lecture course was, besides the witty treatment of the subject in the most apt phraseology, the not infrequently interpolated polemic and many a piquant digression which never failed to refresh the weakened thought of the listeners and to stimulate to new effort.... with straightedge or compasses resting on his knee, his somewhat stooped body in rocking motion—he allowed his humor to stream forth without delay, with undimmed eye, with roguish smiles about the well formed mouth. At the time of which I am speaking his curly hair had already become gray; but a youthful charm still illumined his features, which betrayed even to a stranger his lovable personality. So much the more did one bemoan the isolation in which Thibaut continually remained, far from the social life of the university. In the warm season, as long as the weather allowed, he lived in his cheerful, well cultivated garden at the foot of the Hainberg, where we young people, who enjoyed his favor, liked to visit him Sunday mornings. On fair days we found him secure in the large arbor, which—covered over by a white canvas tent—served as a protector against glare of sunshine and the disturbing drops of a passing rain cloud. Hardly rising from a semi-recumbent position, he then laid aside with a friendly greeting the book whose print betrayed a non-mathematical content, and with the quickest, most skillful turns directed the conversation to objects of general interest, which he was wont to discuss with the gayest humor and sharpest wit.... Art, literature, politics and philosophy furnished him alternately the material for those conversations which became even more attractive by reference to events and persons. Architecture also came into play when he had the German roof of his garden house covered over by an Italian, massive one, in order to get a good view there, on which occasion the derision of several colleagues was most sharply repaid. The richest contribution was furnished by the political events of those years: the decisions at Karlsbad, the revolts in Spain, Naples and Greece, the Congress at Troppau and Laibach, events in which Thibaut apparently took the

liveliest interest. If the conversation turned to the field of philosophy, then there spoke in him the decided follower of Kant, the skeptic, who seemed to have no inclination to agree with the modern systems. It was not easy to talk of mathematics in those conversations unless one of those present pressed the point; and really Thibaut was so far removed from all learned pedantry, so rich in general education, as well as in wit and humor, that it did not occur to him to show off by talking shop. To him his science was burdensome as a topic of conversation, and recreation by traversing other fields of the mind was rightly a necessity. There were some who were thus disappointed in their expectations and criticized loudly his intentional avoidance of a rigorous scientific conversation, his interest in literature and his apparent indifference to newer phenomena in science.... one must not forget that Thibaut devoted three, often four hours a day to the preparation of his fresh vitalizing lectures, which appeared to him as the highest duty of his vocation. After such exertion he might well seek recreation outside the realm of abstraction, in order to refresh himself in the fragrant spirit of spring and poesy."

We have another interesting contemporary comment on Thibaut; C. L. Gerling, then a student in Göttingen, later professor of physics in Marburg, wrote thus to J. F. Pfaff on October 21, 1810:

"The *analysis infinitorum* of Professor Thibaut, which I attended last summer, is being continued *publice* this winter for one hour a week. To be sure, I have learned many new things in it, but must confess to my shame that my knowledge of it still rests on a very weak basis, and that I still miss in it that rigorous coherence which prevails in other parts of mathematics. I attribute the blame to the abundant work which astronomy caused me this summer and prevented me from turning my private study to analysis.... I am planning to attend applied mathematics next winter under Prof. Thibaut.... He proceeds in applied mathematics according to Kästner, omits optics, which he occasionally gives *publice*, from this course, in order to be able to treat the other portions more thoroughly. The course begins here, as is easy to suppose, with Professor Thibaut's pure mathematics, in which he proceeds according to his *Compendium*. Here the number of listeners usually ranges between 50 and 70. Among these there are, as may be easily supposed in such a crowd, many who are really not serious about mathematics, who at the beginning of the semester wanted to fill out another hour, and who grabbed this because mathematics can't do any harm (as one recently said to me), or because friends of this ilk boasted so much about Thibaut's course. His lecture is really, rhetorically and stylistically considered,

as beautiful as one can imagine; for Professor Thibaut speaks from the lectern about like Goethe writes. This lecture and the beautiful and elegant periodic structure, for which he continually strives, is probably the reason why he stays far away from the usual school-type demonstration and proves everything by reasoning which is better adapted to periods. It is not my place as a beginner to pass judgment on such things, but I cannot suppress the modest doubt if this doesn't harm more than help a clear survey and evident insight into the coherence, and if the memory would not be better served by an unperiodic *Euclidean* demonstration. .... Here I don't make such a statement. If some one asks me for advice, I advise him to enroll for Professor Thibaut's course, but to study *Lorenz* on the side. .... Among the pupils of Professor Thibaut, especially those who have not had another teacher previously, but get their first instruction from him, or began privately, there are many who are passionately carried away by him and his methods."

The Teachers' Department

Edited by
JOSEPH SEIDLIN

Remarks on the Law of Cosines

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In all standard texts on trigonometry the law of cosines is presented in connection with the solution of oblique triangles to provide in the main for the two cases: given three sides, and given two sides and the included angle. When the sides of the triangle are numbers not easily squared, the formula proves rather clumsy and is usually replaced by other formulas more convenient for use with logarithmic computation. For that reason the formula is not likely to be emphasized and one or two interesting teaching values, which can be derived from it, may be over-looked. It is the purpose of this note to point out two very simple applications which students seem to like and which help them to recall some previous concepts.

Writing

$$c^2 = a^2 + b^2 - 2ab \cos C$$

and considering a and b constant, side c is a function of angle c, with angle C subject to the restriction that its value is between 0° and 180°. At the lower limit, where $c = 0°$,

$$c^2 = a^2 + b^2 - 2ab \text{ or } c = a - b \text{ (for } a > b).$$

Likewise, for $C = 180°$, $c = a + b$ and from the two results we note

$$a - b < c < a + b,$$

a fact which is stated geometrically as follows: "The sum of two sides of any triangle is greater than the third side, and the difference of any two sides is less than the third side.

Considering the case when angle c is acute, we notice that $a^2 + b^2$ is diminished by the positive quantity, $2ab \cos C$, so that $c^2 < a^2 + b^2$; but $2ab \cos C$ gets smaller as C gets larger, so C increases and, for $C = 90°$, $c^2 = a^2 + b^2$ and we have an old friend, the Pythagorean Theorem.

When Angle C is obtuse it is easy to show that $c^2>a^2+b^2$. It is also worth calling attention to the fact, that, since angle C is surely the largest angle, side c is the greatest side, as this matter seems to escape the students' notice.

Another application arises in connection with the so-called "ambiguous" case, namely, given two sides of a triangle and the angle opposite one of them. The law of cosines seems to provide a proper formula for the solution of this case, for, if sides a and b, and angle A are given, we may write

$$a^2=b^2+c^2-2bc\cos A,$$

an equation which has c as the only unknown quantity. We now proceed to recall the methods of solving a quadratic and, curiously enough, find that the presence of so many letters troubles the students considerably. But, upon solving, we find

$$c=b\cos A\pm\sqrt{a^2-b^2+b^2\cos^2A},$$

which can be written at once in the form

$$c=b\cos A\pm\sqrt{a^2-b^2\sin^2A}.$$

It then seems that the question of roots hinges on the value of the discriminant, $a^2-b^2\sin^2A$. This is zero for $a=b\sin A$, and negative for $a<b\sin A$. For $a>b\sin A$, two solutions seem possible, but we must not overlook that c cannot be negative. If we use the negative square root, c will be zero for $a^2-b^2\sin^2A=b^2\cos^2A$, i. e.

$$a^2=b^2(\sin^2A+\cos^2A)\text{ or }a=b:$$

the value of c would be negative for

$$a^2-b^2\sin^2A>b^2\cos^2A,$$

i. e. $a>b$, so two solutions are possible only if $b\sin A<a<b$. The value of c would also be negative if $b\cos A+\sqrt{a^2-b^2\sin^2A}$ is negative which can occur if angle A is obtuse and $b^2\cos^2A>a^2-b^2\sin^2A$ i. e. $b>a$.

Recapitulating, the equation (and therefore the triangle) has solutions as follows:

a. Two solutions for angle A acute and $b\sin A<a<b$.
b. One solution for angle A acute and either $a=b\sin A$ or $a=b$ or $a>b$.
c. One solution for angle A obtuse and $a>b$.

Thus, we can obtain the complete analysis of the "ambiguous" case from algebraic considerations and, at the same time, furnish a neat illustration of the manner in which the various fields of mathematics supplement one another.

The illustrations given are not particularly novel and any teacher of elementary mathematics can probably think of better ones. The purpose in presenting them at all is to remind teachers that work of this sort is interesting and helpful to students. To fail to recognize opportunities to do some of it is to deny your students an excellent chance to perceive the logical consistency of mathematics.

An Easy Method for Drawing Curves in Polar Coordinates

By A. K. BETTINGER
The Creighton University

The method was developed primarily for its applications in the Differential and Integral Calculus where usually a rough sketch of the curve and the limits within which the curve is defined are all that are needed.

In place of the ordinary pole and polar axis, a set of rectangular coordinates axes is used. The positive $x-$axis coinciding with the polar axis; the origin with the pole. The procedure consists in assigning the values which the function "approaches at the coordinate axes" to radial lines in the plane. The number and position of these lines are determined by the coefficient of the angle.

If the coefficient of the angle is a positive integer n, then each quadrant is divided by radial lines into n equal sections. Let these sections be numbered from 1 to 4 counter-clockwise, repeating if necessary, beginning at the polar axis and returning to it. Since all the trigonometric functions can be reduced to either sine or cosine, or both, a table of limiting values for these two functions will be sufficient. Such a table may easily constructed as follows:

Quadrant	sine
1	0 1
2	1 0
3	0 -1
4	-1 0

Quadrant	cosine
1	1 0
2	0 -1
3	-1 0
4	0 1

The first and second values of the function in each quadrant represent the limiting values of the function in that quadrant for the initial and terminal sides repectively. When the angle θ has an integral multiplier n, the n sections play the role of the quadrants. The method is more

readily applied if r is an explicit function of θ. The addition of another column to the table facilitates the drawing of a large number of different types of curves. This is illustrated in the following figures.

Figures 1 and 2 indicate the method applied to the equation $r = \sin 2\theta$. Since the table values for the first two quadrants are positive, the curve begins at the pole in the first section, attains its maximum value on the radial line bisecting the first quadrant and returns to the pole at the terminal side of the second section. The table values for the third and fourth quadrants are negative and are drawn in the third and fourth "contra-sections" (I have used this term quite successfully in class) respectively. The third "contra-section", for example, is determined by the initial and terminal lines of section three produced through the pole.

Quad.	sine	r
1	0 1	0 1
2	1 0	1 0
3	0 -1	0 -1
4	-1 0	-1 0

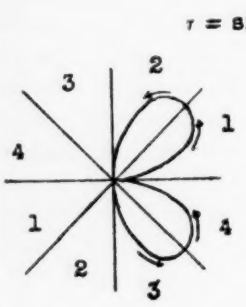


Fig. 1

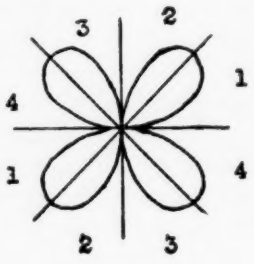


Fig. 2

Quad.	cosine	r
1	1 0	1 0
2	0 -1	0 -1
3	-1 0	-1 0
4	0 1	0 1

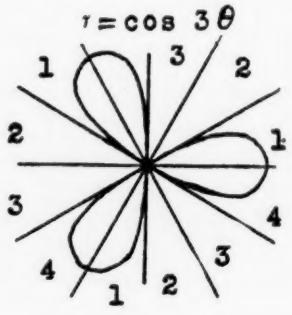


Fig. 3

Quad.	sine	r
1	0 1	0 1
2	1 0	1 0
3	0 -1	0 1
4	-1 0	1 0

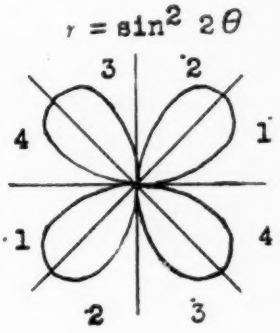


Fig. 4

Quad.	cosine	r
1	1 0	± 1 0
2	0 -1	imaginary
3	-1 0	imaginary
4	0 1	0 ± 1

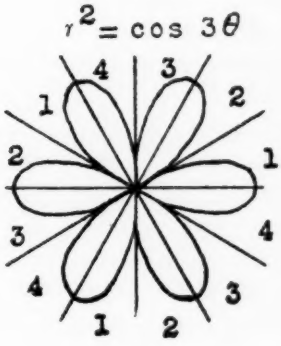


Fig. 5

Quad.	sine	r
1	0 1	1 2
2	1 0	2 1
3	0 -1	1 0
4	-1 0	0 1

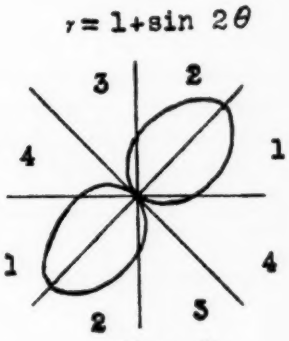


Fig. 6

Quad.	cosine	r
1	1 0	1 3/2
2	0 -1	3/2 3
3	-1 0	3 3/2
4	0 1	3/2 1

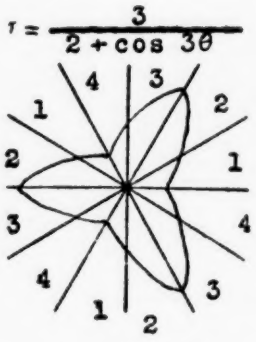


Fig. 7

Let us consider the case where the coefficient of θ is in the form $1/m$, where m is a positive integer. Instead of dividing each quadrant into n sections, m quadrants form one section. For example:

Quadrant	sin
1	0 1
2	1 0
3	0 -1
4	-1 0

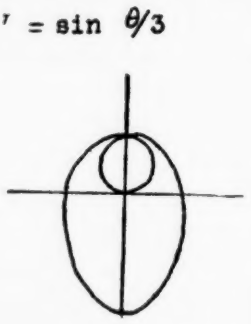


Fig. 8

Since it is necessary for three quadrants to make a section, twelve quadrants must be used to make the four sections required by the table. Then the terminal side of the fourth section will coincide with the polar axis, i. e., the initial side of the first section.

Quadrant	cosine
1	1 0
2	0 -1
3	-1 0
4	0 1

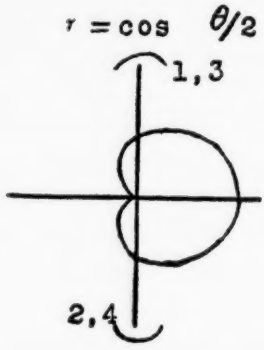


Fig. 9

As above, two quadrants are necessary for one section and hence eight quadrants are used for the complete curve.

When the coefficient of θ is in the form n/m where n and m are both positive integers, the quadrants are divided into n sub-sections and m of these sub-sections are grouped together as a major section for the use of the table. An illustration of this is

Quadrant	sine
1	0 1
2	1 0
3	0 -1
4	-1 0

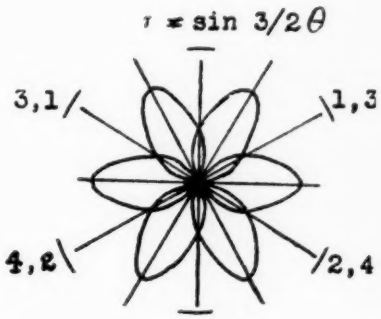


Fig. 10

Obviously then any curve of the type

$$r=\frac{(a+b\sin n'\theta+c\cos n'\theta)^{\alpha}}{(d+e\sin n'\theta+f\cos n'\theta)^{\beta}}$$

can be drawn, where $a, b, c, d, e, f, \alpha, \beta$ may be any positive or negative, fractional or integral numbers and n' may be either n, $1/m$, or n/m, n and m positive integers.

The above formula could be generalized to contain more than one factor in the numerator or the denominator, or both.

A Note on a Certain Trigonometric Equation

By E. C. KENNEDY
The Rice Institute

Consider the equation

(1) $A \sin x + B \cos x = C$, A, B, C real and $A^2+B^2 \geqslant C^2$

Introducing the half-angle and dividing by $\sin^2 x/2$ we get

$$(C-B)\cot^2 x/2 - 2A \cot x/2 + B + C = 0. \qquad \text{Whence}$$

$$\text{(2)} \qquad \cot x/2 = \frac{A \pm \sqrt{A^2+B^2-C^2}}{C-B}, \qquad C \neq B$$

Example: Solve: $4 \sin x + 5 \cos x = 6$.

$$\cot x/2 = 4 \pm \sqrt{5} = 6.2361 \text{ or } 1.7639.$$

$$x/2 = 9^\circ\ 07' + \pi n \text{ or } x/2 = 29^\circ 33' + \pi n$$

$$x = 18^\circ 14' + 2\pi n. \quad x = 59^\circ 06' + 2\pi n.$$

This scheme appears to be much shorter than the usual one. It is not necessary to check a value by substitution in the original equation, since no extraneous roots were introduced. Note that if A, B, C are right triangle numbers it follows that there is only one possible value of x (less than 360°) but in all other cases there are exactly two values, both of which are given by (2). In the special case where $C=B$ we get an indeterminate form. In such a case use

$$\tan x/2 = \frac{A \pm \sqrt{A^2+B^2-C^2}}{B+C} = \frac{A \pm A}{2B}$$

On a Certain Cube Root

By C. S. CARLSON
St. Olaf College

In teaching the solution of the cubic equations, I often find the answers to exercises in a form that indicates that the exercises were made up from predetermined answers.

I have specific reference to the cube root of numbers of the form $a+i\sqrt{b}$ which occur in Cardan's solution of the cubic equation.

If the following note is in the mathematical literature it surely is not well known. I found it of value to the class in simplifying certain of their results to the form of the answers in the texts.

By independent expansions and comparisons we can determine that:

Theorem: If $\sqrt[3]{a+i\sqrt{b}}=x+i\sqrt{y}$ then $\sqrt[3]{a-i\sqrt{b}}=x-i\sqrt{y}$.

Then by cubing, $x^3+3x^2i\sqrt{y}-3xy-iy\sqrt{y}=a+i\sqrt{b}$

gives

$$x^3-3xy=a \tag{1}$$

$$(3x^2-y)\sqrt{y}=\sqrt{b}$$

Now also $x^2+y=\sqrt[3]{a^2+b}$ by multiplying together the conjugates.

Thus if $\sqrt[3]{a^2+b}=c$ we have

$$x^2+y=c. \tag{2}$$

And using (2) and (1) we obtain the cubic

$$x^3-3(c-x^2)=a. \tag{3}$$

If (3) can be solved for a rational x, from (2) we obtain y and $x+i\sqrt{y}$, and the desired cube root has been determined.

Hyperbolic Function Substitution

By HARRY GWINNER
University of Maryland

As early as 1886, Edwards in his differential calculus preface mentioned "that the direct and inverse hyperbolic functions are freely used in the text."

The writer of this article has deemed such substitutions of so much value to the student who is slow in the calculus, that he has fully detailed them in finding the length of a parabolic arc of $y^2=4ax$, $a>0$ from the origin to the end of the latus rectum and because he considers it more instructive, he has taken the more difficult formula for the length viz,

$$S=\int\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

When the $\frac{dy}{dx}$ is substituted in the formula, it becomes

(1) $S=\int\sqrt{\frac{x+a}{x}}\,dx$. Rationalizing the numerator,

(2) $S=\int\sqrt{\frac{x(x+a)}{x}}\,dx$. Substituting the hyperbolic function

for x, then $x=a\sinh^2 u$, $u=$inv. sinh $\sqrt{x/a}$, $dx=2a\sinh u\cosh u\,du$, and $x+a=a(1+\sinh^2 u)$, and (2) becomes

(3) $$S=\int\sqrt{(a\sinh^2 u)a\cosh^2 u}\left(\frac{2\sinh u\cosh u}{\sinh^2 u}\right)du$$

(4) $=2a\int\cosh^2 u\,du$, and for limits when $x=0$, $u=0$, and when $x=a$, $u=$inv. sinh 1, therefore (4) becomes

(5) $$S=a\int_0^{\text{inv. sinh }1}(\cosh 2u+1)du$$

(6) $$= a\left[\frac{\sinh 2u}{2}+u\right]_0^{\text{inv. sinh 1}}$$

(7) $$= a\left[(\sinh u)\ (\cosh u)+u\right]_0^{\text{inv. sinh 1}}$$

(8) $= a[\sqrt{2}+\log(1+\sqrt{2})]$, since inv. sinh $x = \log_e (x+\sqrt{x^2+1})$, which is the result.

Mathematical World News

Edited by
L. J. ADAMS

Dr. E. R. Hedrick, head of the department of mathematics of the University of California at Los Angeles, has been appointed vice-president of the University of California and provost of the University of California at Los Angeles. This position is technically that of administrative head of the University of California at Los Angeles. Dr. Hedrick is a well known figure in the mathematical world, having served as president of the Mathematical Association of America and at present being editor-in-chief of the Bulletin of the American Mathematical Society. In fact Dr. Hedrick's scientific activities are far too numerous to attempt to mention here.

Professor George E. Barnett of the Johns Hopkins University has accepted an invitation from the Carnegie Corporation to make a study of Arbitration Courts in New Zealand and Australia. He will be on leave of absence for the academic year 1937-38.

The annual meeting of the Mathematical Association of England was held in London on January 4-5, 1937. Professor L. N. G. Filon was elected president for the calendar year 1937. The retiring president's address, delivered by Professor A. R. Forsyth, was entitled *Applied Mathematics in School Teaching.*

The English magazine, *Nature,* published weekly by Macmillan and Co. of London, carries in each issue a list of appointments vacant. Applications are invited for the appointments, which include positions in the engineering, scientific and mathematical fields. This magazine also gives space to the publication of the titles of the papers read before the meetings of the scientific academies and societies throughout Europe.

The 15th annual meeting of the Southeastern Section of the Mathematical Association of America is scheduled to be held at Vanderbilt University and George Peabody College, Nashville, Tenn., Friday and Saturday, April 16-17. Dean R. D. Carmichael of the University of Illinois is scheduled to be the guest speaker.

Dr. J. Neyman, of the Department of Applied Statistics, University College, London, is scheduled to be in America from April 5 to May

15. He will lecture on theoretical and applied statistics at Princeton University, the Graduate School of the U. S. Department of Agriculture, the University of Illinois, the State University of Iowa, Iowa State College, and probably in Chicago, New York, and Ann Arbor.

Mathematische Annalen, the well known German journal was founded in 1868 by Alfred Clebsch and Carl Neumann. It is now edited by David Hilbert, with the assistance of Otto Blumenthal, Erich Hecke and Bartel L. van der Waerden, and is published by the firm of Julius Springer in Berlin.

The Southern California Section of the Mathematical Association of America met in Los Angeles on Saturday, March 6, 1937 at the Los Angeles Junior College. Professor W. M. Whyburn was chairman of the program committee, and Dr. S. E. Urner was chairman of the meeting. The following papers were read:

1. *Spherical Harmonics with Spin.* Professor L. J. Adams, Santa Monica Junior College.
2. *Descartes and Three Centuries of Analytic Geometry.* Professor A. D. Michal, California Institute of Technology.
3. *Strong Summability of Sequences.* Dr. Hugh Hamilton, Pomona College.
4. *The Extension of Horner's Method to the Solution of n Equations in n Unknowns.* Professor L. E. Gurney, University of Southern California.
5. *Early Contributors to the Calculus of Variations.* Dr. M. R. Hestenes, University of California at Los Angeles.
6. *Some Auto-Polar Configurations.* Dr. Dewey C. Duncan, Compton Junior College.

The Japanese have entered the field of manufacturing slide rules. The Hemmi slide rules, made of bamboo and of laminated construction, are in common use in the United States. They are manufactured by the Hemmi Seisakusho Co. of Tokio.

The 18th Annual Meeting of the National Council of Teachers of Mathematics was held at Palmer House, Chicago on February 19-20, 1937. Sections included those devoted to Arithmetic, High Schools and Junior Colleges. There were, in addition, two general meetings and a discussion luncheon.

An important source of research memoirs is *Sitzungsberichte der mathematisch-naturwissenschafthchen* of the Bayerischen Akademie der Wissenschaften zu München.

W. Heffer & Sons, Limited, Cambridge, England offer complete sets of English mathematical journals for sale.

Professor Arthur J. Lewis reports that the Rocky Mountain Section of the Mathematical Association of America is to meet at Greeley, Colorado, April 16 and 17, 1937. Colorado State College of Education will serve as host. Dr. A. E. Mallory of that institution is chairman this year.

Dr. David Hilbert, professor of mathematics emeritus at the University of Göttingen, celebrated his 75th birthday on January 23, 1937.

The American Mathematical Society met at Columbia University, New York City, Friday and Saturday, March 26-27, 1937.

Friday afternoon was devoted to a symposium on Laplace Integrals. By invitation of the Program Committee, Professors Einar Hille, J. D. Tamarkin and D. V. Widder gave brief addresses and the discussion was led by Professors Salomon Bochner and H. F. Bohnenblust.

A group of members of the Society arranged a conference on Algebraic Geometry for Thursday, March 25, the day preceding the New York meeting.

The American Mathematical Society met at the University of Chicago on Friday and Saturday, April 9-10, 1937.

Friday afternoon, by invitation of the Program Committee, Dr. W. T. Reid delivered an address on the subject *Boundary value problems of the calculus of variations.*

Problem Department

Edited by
ROBERT C. YATES

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscript be typewritten with double spacing. Send all communications to Robert C. Yates, College Park, Maryland.

PROBLEMS FOR SOLUTION

No. 138. Proposed by M. H. Martin, University of Maryland.

Given a polynomial:

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \qquad (a_n > 0)$$

with integral coefficients. The quantity

$$h = n + a_n + |a_{n-1}| + \cdots + |a_1| + |a_0|$$

is known as the *height* of the polynomial.*

(1) Find a formula for the number of polynomials possessing a given height h.

(2) Find a formula for the number of polynomials possessing a given height h and a given degree n.

(3) Show that for polynomials having a fixed height h the number of polynomials of degree n equals the number of degree $h-n-1$.

No. 139.

Given the triangle ABC. Find points P on AC and Q on BC such that $AP = PQ = QB$.

No. 140. Proposed by Robert C. Yates, University of Maryland.

*As defined by Cantor.

Integrate:

$$\int \tan ax \cdot \tan bx \cdot \tan(a+b)x \cdot dx$$

No. 141. Proposed by Robert C. Yates, University of Maryland.

Given two lines and a circle in a plane. Find that path-triangle of minimum perimeter whose vertices lie on these three elements.†

No. 142. Proposed by Alfred Moessner, Nurnberg—N, Germany.

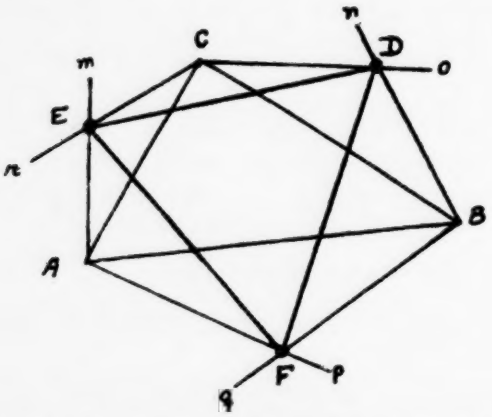
What is the general solution of the identity:

$$xyz = x^2 + y^2 + z^2$$

in integers?

No. 143. Proposed by Kirby W. Blain, Lakeland, Florida.

Given any triangle *ABC* with lines *m*, *n*, *o*, *p*, *q*, *r* drawn at 30° angles with the sides as shown. Prove triangle *DEF* equilateral by elementary methods.



†This is typical of a set of nine problems dealing with points (*P*), lines (*L*), and circles (*C*); i. e., (*PLL*), (*PPL*), (*LLL*),....(*CCC*). The case (*PPC*) is noteworthy. See an article by G. A. Bingley, *American Mathematical Monthly*, October, 1926.

Reviews and Abstracts

Edited by
P. K. SMITH

Die Sphärik von Menelaos aus Alexandrien in der Verbesserung von Abu Nasr Manṣūr b.'Alī b. 'Iraq, mit Untersuchungen zur Geschichte des Textes bei den islamischen Mathematikern. By Max Krause. Abhandlungen der Gesellschaft der Wissenschaften zu Göttingen, phil.-hist. Klasse, 3. Folge: No. 17, (Sonderheft der math.-phys. Klasse), Berlin, Weidmannsche Buchhandlung, 1936. VII+366 pages+7 pages of plates.

This volume will be welcomed by all careful scholars who are interested in the historical study of primary sources. The Greek original text of Menelaus' work on spherical trigonometry has been lost, and our knowledge of it was based principally on the three Latin translations, which in turn came partly by way of Jakob ben Makhir's Hebrew translation from the Arabic. The five Arabic editions have been largely neglected. Menelaus was one of the Greek authors who was translated very early into the Arabic, although the earliest translation is thought to have been into the Syrian. The present edition was dated September 17, 1007, and the scribe's copy of it November 2, 1279.

Dr. Krause has given us the complete Arabic text of Prince Abū Naṣr Manṣūr b. 'Alī b. 'Irāq's edition of Menelaus, which is contained in a manuscript at the University of Leyden. We find also a very clear German translation of the entire text. The most painstaking scholarship is exhibited to us in 116 pages of textual criticism. (Dr. Krause used the best work on Menelaus: A. A. Björnbo, *Studien über Menelaus' Sphärik*, Leipzig, 1902.) A thorough study of the Hebrew, Latin, and Arabic translations is made, and also of their relationship to each other. On the Arabic translators and commentators this is particularly exhaustive. Dr. Krause has skilfully brought together all that we know about Prince Abū Naṣr Manṣūr b. 'Alī b. 'Irāq. It is felt that this Prince's edition is closest to the best Arabic translation, which will explain why it was chosen for this volume. Varying and alternative proofs of theorems in the other Menelaus editions are indicated, as well as interpolations or corrections made by the translators and commentators.

Work on this volume was finished by Dr. Krause in September, 1932, and the reader should feel grateful to Professor Otto Neugebauer who has promoted its publication. It well typifies that sort of scholarly editing of manuscripts constituting source material, which he has repeatedly said is so necessary if we are to know anything accurate about the history of mathematics, especially of the Orient and of antiquity. His own example in this field is stirring others to action. Thanks are due the Royal Society of Sciences in Göttingen and the Hamburg University—Society for bearing the costs of publication.

University of Illinois. G. WALDO DUNNINGTON.

The Teaching of Mathematics. A Source Book and Guide. By Raleigh Schorling. The Ann Arbor Press, Ann Arbor, 1936. VIII+ 247 pages.

The book is divided into five parts with 22 chapter headings followed by a bibliography adequate for extensive reading. The preface indicates that the writer has assembled materials relative to the teaching of secondary mathematics which have been accepted by curriculum makers and good teachers since the beginning of the present century. The compilation is important at present because of its bearing on the question of a New Curriculum in which mathematics would not be taught. Regarding the current claims that mathematics is not useful and that teaching is traditional we find that Professor Schorling has presented us with a set of very annoying quintuplets. We will take a look at them in the order of their appearing.

Part I. Aims, Principles, and General Objectives.

Aims are listed as, utilitarian, disciplinary, and cultural. As minimum achievements we find accuracy and facility in computation, ability to apply fundamental laws of arithmetic and algebra to new situations, judgments of error and approximation, ability to solve problems by use of formulas and equations, graphic interpretation of results, geometric forms and space relations, general mental discipline, analysis, recognition of logical relations, generalization, functional thinking, appreciation of beauty, logical precision, power, and the permanence of laws and relations. He says these should be taught until they will function in the life of the student. He would avoid isolation and mere accumulation of the unrelated. He would require skill in connected thinking as a means to effective use in actual situations which cannot be anticipated in given forms. We list only a sample of the abilities asked for and the writer infers that he has

listed only a part of the desirable aims. If mathematics were abandoned we would abandon the very direct approach to the above objectives in secondary education.

Part II. The Reform Movement in Secondary Mathematics.

Reprints are quoted from the Presidential address of E. H. Moore to the American Mathematical Society, 1902. With reference to the Perry Movement and the demand for practical emphasis at lower levels the plea is made for objective, experimental, measurable, interpretation of mathematics as basis for valid laws and principles of conduct. In pursuit of these values we had a beginning of books on general mathematics. When such books began to appear the writers of arithmetic, algebra, and geometry, began the use of all three in each. Arithmetics contained drawings and formulas, algebras built in numerical solutions and drawings with long lists of problems from human experience, and geometry was full of drawing and numerical equivalents of forms and formulas. The Laboratory was set up by which the teacher makes objective the meaning of things. Teachers are cautioned to give due regard to the history of curriculum developments in mathematics as back ground for wise connection between mathematical principles and actual life. There must be some profound connection between secondary mathematics and life which brought the writer to such a state of enthusiasm regarding opportunity for enrichment of curricula.

Part III. Five Major Tasks of the Mathematics Teacher.

Here we have 47 pages devoted to detailed outlines and instructions regarding the following topics.

1. A list of specific objectives for grades 7, 8, and 9.
2. Teaching the Problem.
3. Methods of developing interest.
4. The new psychology of drill.
5. Measuring achievement.

Herein is effort on a high level to provide material for methods and uses of those methods to practical living ends.

Part IV. Techniques for troublesome spots.

The writer recounts procedure for teaching formulas, equations, graphs, directed numbers—meaning positive and negative—and relationships. A brief teaching procedure is outlined for numerical trigonometry in the junior high school. The program for teaching geometry is divided into Intuitive and Demonstrative. I would prefer to say first teach notions about geometry and then teach geome-

try. The section is concluded by a long outline of good practice in teaching arithmetic. He lists a few of several thousand known uses among which are insurance, budgets, interests, taxes, and volumes of various shapes. Projects, drills, dull pupils, tests and remedies all come in for honorable mention.

Part V. Side Lights on Teaching of Mathematics.

We are urged to make use of the rich and fascinating history of the elementary subjects. Short sketches are given to illustrate. The last chapter is devoted to Mathematical Recreations. The list includes a generous selection of elementary, novel, useful, and amusing problems; mathematics set to music; projects in collecting and arranging; cartoons, field trips, arguments in court, etc.

Each Section is followed by a list of exercises which raise a variety of practical questions for the teacher. The work seems to establish its claim to being a Source Book. We have here a valuable account of the trend of secondary mathematics during the present century. Let us add this one to our list of publications which indicate why mathematics cannot be removed from the secondary school curriculum. The whole work seems to advocate the practical approach to functional values in elementary education. It should be in every library where mathematics is taught in the school.

The copy before me has only a paper binding.

State College, Mississippi. C. D. SMITH.

Triumph der Mathematik. Hundert berühmte Probleme aus zwei Jahrtausenden mathematischer Kultur. By Heinrich Dörrie. Ferdinand Hirt, Breslau, 1933, VII+386 pages.

The object of this work, so the author says in the preface, is to combine the famous problems of elementary mathematics in one book, to point out their origin, and to present their solution in a short, natural, and clear manner. The book was written with special regard for those readers who do not have the time or opportunity to devote themselves to the study of advanced mathematics. In spite of this restriction on its material, the book represents a most fascinating picture of a colorful variety of mathematical methods, which delights any reader who is interested in mathematics, and enjoys seeing examples of unusual mathematical brilliance in reasoning.

In order to keep down the size, and the price of the book, the author selected one hundred problems, which together with their

solution would not take up too much space. The selection of problems is excellent in so far as they represent a mosaic of some of the most striking mathematical achievements of men like Euler, Gauss, Leibniz, and other great mathematicians.

Since a knowledge of analysis, that is of even the calculus, is not presupposed, the expansion of the elementary transcendental functions into series, which is taken up in the book, could not be based on Taylor's theorem. The author however gives a most interesting method for obtaining these series on the basis of the mean value of these functions over an interval. A knowledge of the fundamentals of vector analysis, only a few simple theorems of elementary mathematics, and perhaps also a more or less mathematically mature mind are the only prerequisites for the understanding and enjoyment of the book. The great variety of the problems considered is indicated by the following list taken from the table of contents: *Problema bovinum* of Archimedes, Fermat's equation, the Hermite-Lindemann theorem on transcendental numbers, Pascal's theorem on the hexagon, problems in navigation, and astronomy.

The book is especially useful to undergraduate mathematics clubs, and should be in every college library.

College of St. Thomas. H. P. THIELMAN.

Elements of Probability. By H. Levy and L. Roth. Oxford University Press, London. New York. 1936. x+200 pages. $5.00.

Within recent years interest in Probability has increased greatly. Therefore this new work on the subject has appeared. Announcing that no effort is being made to treat all fields peculiar to their subjects, the authors propose only to produce an elementary text. Probability is "an essential to scientific method"; so it is necessary only to interpret it as "a guide to scientific procedure." Hence the form given the analysis of the several topics treated has been determined by the special reasons governing their respective selection.

In the first chapters of the book, the non-mathematical student is not overlooked: Chapters III and IV, dealing with the *Theory of Arrangements* and *The Elementary Theorems of Mathematical Probability*, are especially for him. The authors have refrained from entering the field of statistics itself except for slight incursions to make clear specific points.

The text contains nine chapters: Chapter I is an eleven-page historical sketch. It is brief, but serves well to arouse general interest.

Chapter II presents the scope of Probability: It defines chance, cites several definitions of Probability, and explains what is meant by "mathematical determinism." Chapters III and IV, mentioned above, cover only fourteen pages.

Bernoulli's theorem, its extension and its relation to the *normal law* occupy the fifteen pages of Chapter V. A proof of Sterling's theorem is included. In chapter VI, entitled *Extension to Continuous Distributions*, Probability is defined both for a one-dimension medium and for a two-dimension medium. Buffon's problem appears as one of the illustrative examples.

The last three of the nine chapters cover more than half of the total text, and, of course, are intended to occupy the more serious attention of the student. They treat, in order: Chapter VII, *The Theory of Arrangements*; Chapter VIII, *The Empirical Theory of Distribution*, including a study of the Gaussian Law and other theorems of hypothetical populations; and Chapter IX, *The Use of Probability in Scientific Induction*. In this last chapter, the student considers the statement of the problem, deduction of a sample, derivation of a particular population, and determination of a hypothetical population.

Illustrative examples and exercises for the student appear throughout the text. "Most of them are new," but some are from Whitman's *Choice and Chance*.

The mechanical features of the book make it very attractive: Cover, 6x9 inches, navy-blue buckram; type, easy on the eye; paragraphing and paragraph headings, well chosen; good arrangement of material on the page; its table of contents and index, carefully prepared.

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